TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 4, April 1998, Pages 1359–1378
S 0002-9947(98)02136-9

COVERS OF ALGEBRAIC VARIETIES III. THE DISCRIMINANT OF A COVER OF DEGREE 4 AND THE TRIGONAL CONSTRUCTION

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Abstract. For each Gorenstein cover $\varrho\colon X \to Y$ of degree 4 we define a scheme $\Delta(X)$ and a generically finite map $\Delta(\varrho)\colon \Delta(X) \to Y$ of degree 3 called the *discriminant of* ϱ. Using this construction we deal with smooth degree 4 covers $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ with $n \geq 5$. Moreover we also generalize the *trigonal construction* of S. Recillas.

1. Introduction and notations

In [C–E] a general structure theorem for Gorenstein covers of degree d has been proved. Such a result has been used for characterizing covers of degree 3, 4 and 5 (see [C–E] and [Cs] respectively) of algebraic varieties. More precisely, in section 4 (resp. 3) of [C–E] it is proved that the total space X of each Gorenstein cover $\varrho\colon X \to Y$ of degree 4 (resp. 3) is obtained inside a suitable projective bundle $\pi\colon \mathbb{P} \to Y$ of rank 2 (resp. 1) as the base locus of a pencil of relative conics (resp. the zero locus of a relative cubic form in two variables), and $\varrho = \pi_{|X}$.

For each pencil of conics $\mathfrak{b}$ in $\mathbb{P}^2_k$ spanned by two quadratic forms $a := x^tAx$ and $b := x^tBx$ (A and B being symmetric 3×3 matrices) the discriminant $\Delta(\mathfrak{b})$ is the cubic (possibly zero) polynomial $\Delta(\mathfrak{b}) = \det(sA + tB)$. The roots of $\Delta(\mathfrak{b})$ correspond to the degenerate conics of $\mathfrak{b}$.

In this paper we investigate the geometry of the Galois–theoretic relationship between the general equations of degree 4 and 3. We are thus able to generalize in arbitrary dimension the *trigonal construction* due to S. Recillas for covers of $\mathbb{P}^1_k$ (see [Re]). We do this globalizing the description above (exploiting an idea due to T. Ekedahl: see [Ek]).

In section 4 we define for each Gorenstein cover $\varrho\colon X \to Y$ of degree 4 its discriminant $\Delta(\varrho)\colon \Delta(X) \to Y$, getting the following result.

Theorem 1.1. *Let Y be an integral scheme defined over an algebraically closed field k of characteristic $p \neq 2$. To each Gorenstein cover $\varrho\colon X \to Y$ of degree* 4 *one can associate its discriminant $\Delta(\varrho)\colon \Delta(X) \to Y$, which is generically finite of degree* 3*. The points $y \in Y$ over which $\Delta(\varrho)$ does not have finite fibres are exactly*

Received by the editors December 1, 1995.
1991 *Mathematics Subject Classification.* Primary 14E20, 14E22.
Key words and phrases. Cover, Gorenstein, discriminant.
This work was done in the framework of the AGE project, H.C.M. contract ERBCHRXCT 940557.

the points such that

$$\varrho^{-1}(y) \cong \operatorname{spec}\left(\frac{k(y)[u,v]}{(u^2,v^2)}\right),$$

and in this case $\Delta(\varrho)^{-1}(y) \cong \mathbb{P}^1_{k(y)}$. *If there are not such* y*'s, then* $\Delta(\varrho)$ *is a Gorenstein cover of degree* 3. □

Moreover we also study the reducedness, connectedness and smoothness of the discriminant cover, giving many examples exhibiting various possible behaviours.

In section 5 we determine the class of $\operatorname{Sing}(\Delta(X))$ under the hypotheses of genericity of $\varrho\colon X \to Y$.

Section 6 is devoted to the generalization of the above mentioned trigonal construction to covers of arbitrary smooth schemes Y satisfying some mild technical conditions (see Theorem 6.3).

In section 7 we deal with covers of $\mathbb{P}^n_{\mathbb{C}}$.

Theorem 1.2. *If* X *is integral and smooth,* $n \geq 5$*, and* $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ *is a cover of degree* 4 *such that* $\varrho_*\mathcal{O}_Y/\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$ *is uniform, then* $\varrho_i^*\colon H^i(\mathbb{P}^n_{\mathbb{C}},\mathbb{C}) \to H^i(X,\mathbb{C})$ *is an isomorphism for* $i \leq 0 \leq n-1$ *and a monomorphism (but not an isomorphism) for* $i = n$. □

Taking into account Fujita's description of covers $\varrho\colon X \to Y$ of degree 3 with both X and Y smooth (see [Fj]), we are also able to give a qualitative study of the locus of points of total ramification of a cover of degree 4.

Theorem 1.3. *If* $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ *is a cover of degree* 4 *with* X *smooth and* $n \geq 5$*, then* X *is a quadrisecant of an ample line bundle if and only if for each* $y \in \mathbb{P}^n_{\mathbb{C}}$ *there is an embedding* $X_y := \varrho^{-1}(y) \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$. *More precisely, either* X *is a quadrisecant of an ample line bundle, or there is a point* $y \in \mathbb{P}^n_{\mathbb{C}}$ *such that the fibre of* ϱ *over* y *is isomorphic to*

$$\varrho^{-1}(y) \cong \operatorname{spec}\left(\frac{\mathbb{C}[u,v]}{(u^2,v^2)}\right).$$

In this case y *has at least multiplicity* 4 *in the branch locus of* ϱ. □

Now we recall some definitions, notations and results which will be used in what follows.

A local ring R is Cohen–Macaulay if $\dim(R) = \operatorname{depth}(R)$. A Cohen–Macaulay ring R is called Gorenstein if its injective dimension $id_R(R)$ is finite. An arbitrary ring R is called Cohen–Macaulay (resp. Gorenstein) if $R_{\mathfrak{M}}$ is Cohen–Macaulay (resp. Gorenstein) for every maximal ideal $\mathfrak{M} \subseteq R$.

In this paper all schemes are assumed to be noetherian. All schemes over a field k of characteristic p are assumed to be separated and of finite type over k, and we always take $p \neq 2$.

A scheme X is Cohen–Macaulay (resp. Gorenstein) if for each point $x \in X$ the local ring $\mathcal{O}_{X,x}$ of X in x is Cohen–Macaulay (resp. Gorenstein). If X is defined over k, then it is Gorenstein if and only if it is Cohen–Macaulay and its dualizing sheaf $\omega_{X|k}$ is invertible.

Let Y be a scheme. A flat morphism $\varrho\colon X \to Y$ is said to be Gorenstein of relative dimension r if for each $y \in Y$ the scheme–theoretic fibre $X_y := \varrho^{-1}(y)$ is a Gorenstein scheme over $k(y)$ of dimension r. In particular, the relative dualizing sheaf is defined and invertible.

Let Y be an scheme. A morphism $\varrho\colon X \to Y$ is called a cover of degree d if $\varrho_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$–sheaf of rank d: ϱ is a cover if and only if it is flat and finite. If Y is smooth and X is Cohen–Macaulay, then every finite surjective morphism is a cover.

There exists an exact sequence of the form $0 \to \mathcal{O}_Y \to \varrho_*\mathcal{O}_X \to \check{\mathcal{E}} \to 0$, where $\check{\mathcal{E}}$ is a locally free $\mathcal{O}_Y$–sheaf of rank $d-1$ called *the Tschirnhausen module of* ϱ. If the characteristic p of k does not divide d, then the above sequence splits and $\varrho_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \check{\mathcal{E}}$ (generalize the proof of lemma 2.2 in [Mi]). If the cover $\varrho\colon X \to Y$ is Gorenstein, then $\big(\varrho_*\mathcal{O}_X\big)\check{} \cong \varrho_*\,\omega_{X|Y}$ (see [Ha], exercise III 6.10), and hence $\varrho_*\,\omega_{X|Y} \cong \mathcal{O}_Y \oplus \mathcal{E}$.

If $\mathcal{E}$ is a locally free $\mathcal{O}_Y$–sheaf of rank $d+1$, we denote by $\pi\colon \mathbb{P}(\mathcal{E}) \to Y$ the corresponding projective bundle, i.e. $\mathbb{P}(\mathcal{E}) := \mathbf{Proj}(\mathcal{S}\,\mathcal{E})$ ($\mathcal{S}\,\mathcal{E}$ denotes the symmetric algebra of $\mathcal{E}$, $\mathcal{S}^n\,\mathcal{E}$ its component of degree n), and π is induced by the natural monomorphism $\mathcal{O}_Y \rightarrowtail \mathcal{S}\,\mathcal{E}$ (see [Ha]).

If $\eta \in H^0\big(Y, \mathcal{F}\big)$ is a section of a locally free $\mathcal{O}_Y$–sheaf $\mathcal{F}$, we denote by $D_0(\eta)$ its zero–locus.

I would like to express my thanks to F. Catanese for a number of helpful discussions and interesting suggestions.

2. Covers of degree 3 and 4

For the following results we refer to sections 3 and 4 of [C–E]. Let X and Y be schemes, Y integral, and let $\varrho\colon X \to Y$ be a Gorenstein cover of degree $d=3$ or $d=4$. There exists a natural factorization $\varrho = \pi \circ i$, where $\pi\colon \mathbb{P} \to Y$ is the natural projection, $\mathbb{P} := \mathbb{P}(\mathcal{E})$, $\check{\mathcal{E}}$ being the Tschirnhausen module of ϱ and $i\colon X \hookrightarrow \mathbb{P}$ is a closed embedding. Moreover, $\omega_{X|Y} \cong \mathcal{O}_{\mathbb{P}}(1)_{|X}$, and for each $y \in Y$ the fibre $X_y := \varrho^{-1}(y)$ is a complete intersection in $\mathbb{P}_y := \pi^{-1}(y)$. If $d=3$ then $X_y \subseteq \mathbb{P}_y \cong \mathbb{P}^1_{k(y)}$ is the zero locus of a single cubic polynomial, whereas if $d=4$ then $X_y \subseteq \mathbb{P}_y \cong \mathbb{P}^2_{k(y)}$ is the base locus of a pencil of conics without fixed components.

If $d=3$ there is a unique exact sequence

$$0 \to \pi^*\det\mathcal{E}(-3) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0.$$

If $d=4$ there is a unique locally free $\mathcal{O}_Y$–sheaf $\mathcal{F}$ of rank 2 such that $\det\mathcal{F} \cong \det\mathcal{E}$ and an exact sequence

$$0 \to \pi^*\det\mathcal{E}(-4) \to \pi^*\mathcal{F}(-2) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0. \tag{2.1}$$

In particular, in both the above cases, $X = D_0(\delta)$. Using the natural isomorphisms

$$\begin{aligned} \Phi_3\colon H^0\big(Y, \mathcal{S}^3\,\mathcal{E} \otimes \det\mathcal{E}^{-1}\big) &\xrightarrow{\sim} H^0\big(\mathbb{P}, (\pi^*\det\mathcal{E}^{-1})(3)\big), \\ \Phi_4\colon H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big) &\xrightarrow{\sim} H^0\big(\mathbb{P}, (\pi^*\check{\mathcal{F}})(2)\big), \end{aligned} \tag{2.2}$$

one gets sections $\eta := \Phi_3^{-1}(\delta)$ or $\eta := \Phi_4^{-1}(\delta)$. Note that $\dim(D_0(\delta_y)) = 0$ for each point $y \in Y$. In this case we say briefly that η *has the right codimension at* $y \in Y$.

Theorem 2.3. *Let Y be an integral scheme. Any Gorenstein cover $\varrho\colon X \to Y$ of degree 3 such that $\check{\mathcal{E}} \cong \operatorname{coker}\varrho^{\#}$ determines, up to scalars,*

$$\eta \in H^0\big(Y, \mathcal{S}^3\,\mathcal{E} \otimes \det\mathcal{E}^{-1}\big)$$

having the right codimension at every $y \in Y$.

Conversely, given $\eta \in H^0\big(Y, \mathcal{S}^3\,\mathcal{E} \otimes \det \mathcal{E}^{-1}\big)$ *having the right codimension at every* $y \in Y$*, let* $X := D_0\big(\Phi_3(\eta)\big) \subseteq \mathbb{P}$*. Then the restriction* ϱ *of the canonical map* $\pi\colon \mathbb{P} \to Y$ *to* X *is a Gorenstein cover of degree* 3 *such that* $\check{\mathcal{E}} \cong \operatorname{coker} \varrho^{\#}$*.*

Proof. See [C–E], theorem 3.4. □

Theorem 2.4. *Let* Y *be an integral scheme. As explained above, any Gorenstein cover* $\varrho\colon X \to Y$ *of degree* 4 *such that* $\check{\mathcal{E}} \cong \operatorname{coker} \varrho^{\#}$ *determines a locally free* $\mathcal{O}_Y$*–sheaf* $\mathcal{F}$ *of rank* 2 *with* $\det \mathcal{F} \cong \det \mathcal{E}$ *and, up to scalars,* $\eta \in H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big)$ *having the right codimension at every* $y \in Y$*.*

Conversely, given a locally free $\mathcal{O}_Y$*–sheaf* $\mathcal{F}$ *of rank* 2 *with* $\det \mathcal{F} \cong \det \mathcal{E}$ *and* $\eta \in H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big)$ *having the right codimension at every* $y \in Y$*, let* $X := D_0\big(\Phi_4(\eta)\big) \subseteq \mathbb{P}$*. Then the restriction* ϱ *of the canonical map* $\pi\colon \mathbb{P} \to Y$ *to* X *is a Gorenstein cover of degree* 4 *such that* $\check{\mathcal{E}} \cong \operatorname{coker} \varrho^{\#}$ *and* $\mathcal{F} \cong \ker\big(\mathcal{S}^2\,\mathcal{E} \to \varrho_*\omega^2_{X|Y}\big)$*.*

Proof. See [C–E], theorem 4.4. □

It is helpful to know when sections having the right codimension at each point exist.

Theorem 2.5. *Let* Y *be a projective, smooth, connected scheme over a field* k*,* $\mathcal{E}$ *and* $\mathcal{F}$ *be locally free* $\mathcal{O}_Y$*–sheaves of ranks* 3 *and* 2 *respectively,* $\det \mathcal{E} \cong \det \mathcal{F}$*, and define* $\mathcal{H} := \mathcal{S}^2\,\mathcal{E} \otimes \check{\mathcal{F}}$*. The sets*

$$H_{rc} := \{\eta \in H^0\big(Y, \mathcal{H}\big) |\ \eta \text{ has the right codimension at every } y \in Y\},$$
$$H_s := \{\eta \in H_{rc} |\ D_0(\Phi_4(\eta)) \subseteq \mathbb{P} \text{ is smooth}\}$$

are open (but possibly empty). If $k = \mathbb{C}$*,* $\dim(Y) \leq 3$ *and* $\mathcal{H}$ *is globally generated, then* $H_s \neq \emptyset$*.* $D_0(\Phi_4(\eta))$ *is connected for any* $\eta \in H_{rc}$ *if and only if* $h^0\big(Y, \check{\mathcal{E}}\big) = 0$*.*

Proof. See [C–E], theorem 4.5. See also [C–E], theorem 3.6, for the degree 3 case. □

3. The branch locus of a cover of degree 4

Let $\varrho\colon X \to Y$ be a Gorenstein cover of degree 4 with invariants $\mathcal{E}$, $\mathcal{F}$ and defined by a section $\eta \in H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big)$. From now on we will denote by $\mathbb{P} := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Y$, $\overline{\mathbb{P}} := \mathbb{P}(\mathcal{F}) \xrightarrow{\overline{\pi}} Y$ the canonical projections.

According to the results of section 2, for each $y \in Y$ we can define the Segre number of X_y, denoted by $S(X_y)$, as the Segre number of the pencil of conics of $\mathbb{P}_y \cong \mathbb{P}^2_{k(y)}$ cutting out the scheme X_y (see [Wa]). We have the Table 3.1 (see [Wa], table 0).

Definition 3.2. Let $\varrho\colon X \to Y$ be a Gorenstein cover of degree 4 and $y \in Y$. We say that ϱ is planar (resp. even) over y if $S(X_y) = [1, 1; ; 1]$ (resp. $S(X_y) = [(1, 1), 1],\ [(2, 1)]$) and we define the two sets $R_{planar}(\varrho) := \{x \in X |\ \varrho$ is planar at $\varrho(x)\}$ (resp. $R_{even}(\varrho) := \{x \in X |\ \varrho$ is even at $\varrho(x)\}$.

Let $\mathcal{G} := \overline{\pi}^*\mathcal{E}$. Since $\mathcal{F} \cong \check{\mathcal{F}} \otimes \det \mathcal{F}$ and $\det \mathcal{F} \cong \det \mathcal{E}$, there is an isomorphism $\Psi\colon H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big) \to H^0\big(\overline{\mathbb{P}}, \mathcal{S}^2\,\mathcal{G} \otimes \det \mathcal{G}^{-1}(1)\big)$. In the following we will identify $\vartheta := \Psi(\eta)$ with the corresponding symmetric map $\vartheta\colon \check{\mathcal{G}}(-1) \to \mathcal{G} \otimes \det \mathcal{G}^{-1}$. Its adjoint $\operatorname{adj}(\vartheta)\colon \mathcal{G} \to \check{\mathcal{G}}(2)$ is symmetric too. By applying $\overline{\pi}_*$ we get

$$\overline{\eta} \in \operatorname{Hom}_{\mathcal{O}_Y}\big(\mathcal{S}^2\,\mathcal{E}, \mathcal{S}^2\,\mathcal{F}\big) \cong \operatorname{Hom}_{\mathcal{O}_{\overline{\mathbb{P}}}}\big(\mathcal{S}^2\,\mathcal{G}, \mathcal{O}_{\overline{\mathbb{P}}}(2)\big).$$

Table 3.1

	$S(X_y)$	$X_y \subseteq \mathbb{P}^2_{k(y)}$
i)	$[1,1,1]$	$V_+(u^2+w^2, v^2+w^2)$
ii)	$[2,1]$	$V_+(uv, v^2+w^2)$
iii)	$[(1,1),1]$	$V_+(u^2+v^2, w^2)$
iv)	$[3]$	$V_+(v^2+2uw, vw)$
v)	$[(2,1)]$	$V_+(w^2+2uv, v^2)$
vi)	$[1,1;;1]$	$V_+(u^2, v^2)$

Lemma 3.3. *$\overline{\eta}$ has rank 1 (resp. 2) at y if and only if ϱ is planar (resp. even) over y.*

Proof. We treat only case i) of Table 3.1, the other ones being similar. In this case $\mathcal{H}_{|\mathbb{P}_y} \cong \big(k(y)[s,t]^{\oplus 3}\big)^{\sim}$ and ϑ_y corresponds to the morphism $k(y)[s,t](-1)^{\oplus 3} \to k(y)[s,t]^{\oplus 3}$ given, up to a suitable linear transformation, by the matrix

$$\begin{pmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & s+t \end{pmatrix}.$$

Therefore it is easy to check that $\overline{\eta}_y\colon\ \mathcal{S}^2\, k(y)^{\oplus 3} \to \mathcal{S}^2\, k(y)^{\oplus 2}$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which has rank 3. □

Suppose that $Y \cong \operatorname{spec}(A)$, where A is local with maximal ideal $\mathfrak{M}$ (y the corresponding point of Y), $\mathcal{E} \cong A^{\oplus 3}$ with basis $\{u,v,w\}$, $\mathcal{F} \cong A^{\oplus 2}$, and let $\varrho\colon X \to Y$ be a Gorenstein cover of degree 4. Theorem 2.4 implies that X is given inside $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}^2_A$ by two quadratic forms $a, b \in A[u,v,w]$. On the other hand, X is affine; thus we can assume that $X_y \cap V_+(w) = \emptyset$. This means that a and b have no common factors modulo $(w)+\mathfrak{M}$; hence we can assume that $a = u^2 +$ other terms in u, v, w and $b = v^2 +$ other terms in u, v, w. Finally, with proper linear transformations one easily gets that $X = \operatorname{spec}(R)$, where

$$R := \frac{A[u,v]}{(u^2+2\alpha v+\beta,\, v^2+2\gamma u+\delta)} \tag{3.4}$$

for suitable $\alpha, \beta, \gamma, \delta \in A$.

If we set $L_z(y) := zy$ for $z \in R$ and $Q\colon R \times R \to A$ is the bilinear form defined by $Q(x,y) := Tr(xy) := Tr(L_{xy})$, then an equation of B_ϱ is $b = \det(Q)$ (see [A–K], proposition 6.6). By (3.4), $\{1,u,v,uv\}$ is a basis of R over A; then

$$\begin{aligned} b &= \begin{vmatrix} 4 & 0 & 0 & 12\alpha\gamma \\ 0 & -4\beta & 12\alpha\gamma & 8\alpha\delta \\ 0 & 12\alpha\gamma & -4\delta & 8\beta\gamma \\ 12\alpha\gamma & 8\alpha\delta & 8\beta\gamma & 48\alpha^2\gamma^2+4\beta\delta \end{vmatrix} \\ &= -64(27\alpha^4\gamma^4 - 18\alpha^2\beta\gamma^2\delta - 4\beta^3\gamma^2 - 4\alpha^2\delta^3 - \beta^2\delta^2). \end{aligned} \tag{3.5}$$

If $X := \operatorname{spec}(R)$ where R is defined by 3.4, then ϑ is given by

$$\begin{pmatrix} s & 0 & \gamma t \\ 0 & t & \alpha s \\ \gamma t & \alpha s & \beta s+\delta t \end{pmatrix}.$$

It follows that $\overline{\eta}$ is induced by

$$\begin{pmatrix} -\alpha^2 & 0 & 0 & \beta & -\alpha & 0 \\ \beta & \alpha\gamma & 0 & \delta & 0 & 1 \\ \delta & 0 & -\gamma & -\gamma^2 & 0 & 0 \end{pmatrix},$$

which has rank less than 3 at y if and only if $\alpha\gamma$, $\alpha\delta$, $\beta\gamma$, $\beta\delta$ are all zero at y. More precisely, up to a permutation of α, β, γ and δ the only distinct cases are $\alpha,\beta,\gamma,\delta \in \mathfrak{M}$; then $S(X_y) = [1,1;;1]$, $\alpha \notin \mathfrak{M}$, $\beta,\gamma,\delta \in \mathfrak{M}$, and then $S(X_y) = [(2,1)]$, and $\alpha,\beta \notin \mathfrak{M}$ and $\gamma,\delta \in \mathfrak{M}$. Then $S(X_y) = [(1,1),1]$.

With the above considerations in mind let us describe the singularities of B_ϱ, using (3.5) and Lemma 3.3 above.

Definition 3.6. Let Y be smooth and integral and let $B \subseteq Y$ be a divisor through a point $y \in Y$.

Let $\mathfrak{M} \subseteq A := \mathcal{O}_{Y,y}$ be the maximal ideal. We say that y is a cuspidal double point (resp. a transverse double point) if the local equation of the tangent cone at B around y is ℓ^2 (resp. $\ell_1\ell_2$) where $\ell \in \mathfrak{M} \setminus \mathfrak{M}^2$ (resp. $\ell_1, \ell_2 \in \mathfrak{M} \setminus \mathfrak{M}^2$ are transversal).

A cuspidal double point is called ordinary if $(\ell, b) \subseteq \mathfrak{M}^3 \setminus \mathfrak{M}^4$, non–ordinary otherwise.

Assume that both X and Y are smooth over a point $y \in Y$.

If $S(X_y) = [2,1]$, ϱ is either étale or analytically a double cover. Therefore B_ϱ must be smooth.

If $S(X_y) = [3]$, ϱ is either étale or analytically a totally ramified triple cover. Therefore (see [Mi], section 4) B_ϱ has at least a double point at y. If y is exactly a double point then it is cuspidal (and, in general, ordinary).

If $S(X_y) = [(1,1),1]$ then we can assume $\alpha,\beta \notin \mathfrak{M}$ and $\gamma,\delta \in \mathfrak{M}$ in (3.4). If $\gamma,\delta \in \mathfrak{M} \setminus \mathfrak{M}^2$ are not proportional the tangent cone at y is $4\beta^3\gamma^2 + \beta^2\delta^2$, and so y is a transverse double point. If either γ and δ are proportional or at least one of them belongs to $\mathfrak{M}^2$, then it is easy to check by direct computation that y is a non–ordinary cuspidal double point.

If $S(X_y) = [(2,1)]$, then we can assume $\alpha \notin \mathfrak{M}$ and $\beta,\gamma,\delta \in \mathfrak{M}$. In this case y is at least a triple point.

Finally, if $S(X_y) = [1,1;;1]$, then $\alpha,\beta,\gamma,\delta \in \mathfrak{M}$. In this case y is at least a fourfold point.

4. The Discriminant of a cover of degree 4

Let $\mathcal{E}$, $\mathcal{F}$ be locally free $\mathcal{O}_Y$–sheaves of ranks 3 and 2 respectively, and such that $\det \mathcal{E} \cong \det \mathcal{F}$. As in section 3, we consider the isomorphism Ψ, and we identify $\vartheta := \Psi(\eta)$ with a symmetric map $\check{\mathcal{G}}(-1) \xrightarrow{\vartheta} \mathcal{G} \otimes \det \mathcal{G}^{-1}$. Taking the determinant of ϑ and applying the projection formula, we finally obtain a map of sets

$$\Delta\colon H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big) \to H^0\big(Y, \mathcal{S}^3\,\mathcal{F} \otimes \det \mathcal{F}^{-1}\big)$$

(recall that $\check{\mathcal{F}} \otimes \det \mathcal{F} \cong \mathcal{F}$).

Let Φ_3 be as in (2.2).

Definition 4.1. Let $\varrho\colon X \to Y$ be a Gorenstein cover of degree 4 corresponding to a section $\eta \in H^0\big(Y, \mathcal{S}^2\,\mathcal{E} \otimes \check{\mathcal{F}}\big)$. The scheme $\Delta(X) := D_0\big(\Phi_3(\Delta(\eta))\big) \subseteq \overline{\mathbb{P}}$ (resp. the map $\Delta(\varrho) := \overline{\pi}_{|\Delta(X)})$ is called the discriminant scheme of X (resp. the discriminant map of ϱ).

If $X = \operatorname{spec}(R)$, where R is defined by 3.4, then

$$\Delta(X) \cong \operatorname{proj}\left(\frac{A[s,t]}{\alpha^2 t^3 - \beta t^2 s - \delta t s^2 + \gamma^2 s^3}\right) \subseteq \mathbb{P}^1_A. \tag{4.2}$$

Note that either $\Delta(X_y) \subseteq \mathbb{P}^1_{k(y)}$ is a subscheme of length 3, or $\Delta(X_y) \cong \mathbb{P}^1_{k(y)}$. This second case is characterized by the vanishing of $\alpha, \beta, \gamma, \delta$ at the point y. In particular, $\Delta(X_y) \cong \mathbb{P}^1_{k(y)}$ if and only if

$$X_y \cong \operatorname{spec}\left(\frac{k(y)[u,v]}{(u^2, v^2)}\right).$$

Remark 4.3. We identify $\Delta(\eta)$ with a map $\mathcal{S}^3\,\check{\mathcal{F}} \to \det \check{\mathcal{F}}$, and

$$R_{planar}(\varrho) = \varrho^{-1}\big(D_0(\Delta(\eta))\big)$$

by the above description. Thus $R_{planar}(\varrho)$ is closed, and either $\operatorname{codim}_X\big(R_{planar}(\varrho)\big) \leq 4$ or $R_{planar}(\varrho) = \emptyset$, since $\operatorname{rk}\big(\mathcal{S}^3\,\mathcal{F} \otimes \det \mathcal{F}^{-1}\big) = 4$.

Proposition 4.4. *Let $\varrho\colon X \to Y$ be a Gorenstein cover of degree 4 with Y integral and invariants $\mathcal{E}$ and $\mathcal{F}$. Then the following assertions are equivalent.*

i) $\Delta(\varrho)\colon \Delta(X) \to Y$ is a cover (of degree 3).

ii) $\Delta(\varrho)\colon \Delta(X) \to Y$ is a Gorenstein cover.

iii) $R_{planar}(\varrho) = \emptyset$.

iv) For each $y \in Y$ the Zariski tangent space at $x \in X_y$ has dimension ≤ 1.

Moreover, if $\Delta(\varrho)$ is a cover its Tschirnhausen module is $\check{\mathcal{F}}$, and if Y is smooth at y the branch loci of $\Delta(\varrho)$ and ϱ, $B_{\Delta(\varrho)}$ and B_ϱ, coincide as divisors around y.

Proof. i) and ii) are trivially equivalent since $\Delta(X) \subseteq \mathbb{P}(\mathcal{F})$ is a divisor. $\Delta(\varrho)$ is not a cover if and only if there is $y \in Y$ such that $\Delta(X_y)$ is not finite; hence i) and iii) are equivalent by the above description. The Zariski tangent space T_x at $x \in X_y$ is the intersection of the Zariski tagent spaces of all the conics of $\mathbb{P}_y$ through X_y. Hence T_x has dimension 2 if and only if all the conics through X_y are singular at x, i.e. if and only if ϱ is planar over y.

The equality $B_{\Delta(\varrho)} = B_\varrho$ must be checked locally; hence we assume $Y = \operatorname{spec}(A)$ and $X \cong \operatorname{spec}(R)$, where R is defined by (3.4). Thus $B_\varrho = \operatorname{div}(b)$, where b is defined by (3.5). On the other hand, if $\Delta(\varrho)$ is a cover we can also suppose that α does not vanish on Y. In this case b coincides, up to the invertible factor α^2, with the discriminant of the polynomial $p(t) := \alpha^2 t^3 - \beta t^2 - \delta t + \gamma^2$ representing $\Delta(X)$ in $\operatorname{spec}(A[t])$. □

From now on we will assume both X and Y are smooth. With these hypotheses we give some results about the reducibility, connectedness and smoothness of $\Delta(X)$.

First of all, note that there is a natural structure of regular conic bundle having $\Delta(X)$ as degeneration divisor (see [Sa], definitions 1.1 and 1.4), $p\colon Bl_X\mathbb{P} \to \overline{\mathbb{P}}$, defined in the following way. If $x \in \mathbb{P} \setminus X$, let $y := \pi(x)$ and let C_x be the unique conic inside $\mathbb{P}_y$ containing both x and X_y. Then C_x corresponds to a point of $\overline{\mathbb{P}}_y$.

Proposition 4.5. *Let $\varrho\colon X \to Y$ be a cover of degree 4 with X smooth and Y smooth and integral. Then:*

i) $\Delta(X)$ is reduced;

ii) if $\Delta(X)$ is reducible there is an invertible $\mathcal{O}_Y$–sheaf $\mathcal{M}$ and an epimorphism $\mathcal{F} \twoheadrightarrow \mathcal{M}$;

iii) $\Delta(X)$ is singular at x over $y \in Y$ if and only if $y \in R_{even}(\varrho) \cup R_{planar}(\varrho)$. x corresponds to the conics of rank 1 cutting out $X_y \subseteq \mathbb{P}_y$. x is a Gorenstein double point with tangent cone of rank at least 2 (briefly, a pseudo–node).

Proof. The statements i) and iii) follow from the above description, [Sa], proposition 1.8, corollary 1.9, and the proof of proposition 1.2 iii) in [Bea].

Assume that $\Delta(X) = Z \cup Z'$. Both Z and Z' are Gorenstein (they are divisors inside $\overline{\mathbb{P}}$), and since the degree of $\Delta(\varrho)$ is 3 we can suppose that $\sigma := \Delta(\varrho)_{|Z}\colon Z \to Y$ is generically finite of degree 1; hence $Z \subseteq \overline{\mathbb{P}}$ is a unisecant divisor. This implies the existence of an epimorphism $\mathcal{F} \twoheadrightarrow \mathcal{M}$ where $\mathcal{M}$ is invertible. □

The following example shows that in Proposition 4.5 we cannot hope to prove the irreducibility of $\Delta(X)$.

Example 4.6. Let Y be an integral smooth scheme, projective over $\mathbb{C}$, $\mathcal{L} \in \operatorname{Pic}(Y)$ globally generated, $\mathcal{F} := \mathcal{L}^2 \oplus \mathcal{L}^2$, $\mathcal{E} := \mathcal{L} \oplus \mathcal{L} \oplus \mathcal{L}^2$. Let $u, v \in H^0\big(Y, \mathcal{E} \otimes \mathcal{L}^{-1}\big) \cong H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{L}^{-1}\big)$, $w \in H^0\big(Y, \mathcal{E} \otimes \mathcal{L}^{-2}\big) \cong H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{L}^{-2}\big)$ (resp. $s, t \in H^0\big(Y, \mathcal{F} \otimes \mathcal{L}^{-2}\big) \cong H^0\big(\overline{\mathbb{P}}, \mathcal{O}_{\overline{\mathbb{P}}}(1) \otimes \overline{\pi}^*\mathcal{L}^{-2}\big)$) be independent sections. For a general choice of $\beta, \delta \in H^0\big(Y, \mathcal{L}^2\big)$, $\gamma \in H^0\big(Y, \mathcal{L}\big)$ the subscheme $X \subseteq \mathbb{P}(\mathcal{E})$ defined by

$$\eta := (u^2 + \beta w^2, v^2 + 2\gamma uw + \delta w^2) \in H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big)$$

is smooth and connected, since $h^0\big(Y, \mathcal{L}^{-1}\big) = 0$. On the other hand $\Delta(X) \subseteq \mathbb{P}(\mathcal{F})$ has the equation

$$p(s, t) := t(\beta s^2 + \delta st - \gamma^2 t^2),$$

and hence $\Delta(X)$ is reducible.

Corollary 4.7. *Let ϱ, X and Y be as above. If B_ϱ is reduced and $\Delta(X)$ is integral, then it is also normal. In particular, if $\dim(Y) = 2$, then $\Delta(X)$ has at most isolated normal double points as singularities.*

Proof. Since $\operatorname{Sing}(\Delta(X))$ lies exactly over $\operatorname{Sing}(B_\varrho)$ (see section 2) if B_ϱ is reduced, then $\operatorname{Sing}(\Delta(X))$ has codimension at least 2. On the other hand, $\Delta(X)$ is Cohen–Macaulay. Hence the statement follows from Proposition 4.5 and [Sa], Proposition 1. □

From now on we will assume that $\Delta(\varrho)$ is a cover, i.e., that $R_{planar}(\varrho) = \emptyset$.

Definition 4.8. Let Y be an integral scheme and $Q \xrightarrow{q} Y$ a conic bundle. We define

$$\operatorname{Sing}(q) := \{x \in Q |\ q \text{ is not smooth at } x\} = \{x \in Q |\ x \in \operatorname{Sing}\big(q^{-1}(q(x))\big)\}.$$

$\operatorname{Sing}(q)$ is closed in Q, and $q(\operatorname{Sing}(q))$ is exactly the discriminant curve of the conic bundle Q. Note that $\operatorname{Sing}(q) \neq \operatorname{Sing}(Q)$.

Now assume that $\Delta(X)$ is not connected. Then $\Delta(X) = Z \cup Z'$ is a reduced scheme with $Z \cap Z' = \emptyset$. We can suppose that $\sigma := \Delta(\varrho)_{|Z}\colon Z \to Y$ is quasi–finite of degree 1, so that it is a cover of degree 1 (see [Ha], exercises III 10.9 and III

11.2), i.e., an isomorphism. In particular, σ^{-1} is a section of $\overline{\pi}$. It follows that there exists a conic bundle $\mathbb{P} \supseteq Q \xrightarrow{q} Y$ containing X such that $Q_y := Q \cap \mathbb{P}_y$ is a conic of rank 2 for each $y \in Y$, and $\mathrm{Sing}(q) \cap X = \emptyset$ (recall that if $\mathfrak{b} \subseteq \mathbb{P}^n_k$ is a pencil of conics, then $C \in \mathfrak{b}$ corresponds to a simple root of the discriminant if and only if $\mathrm{Sing}(C) \cap Bs(\mathfrak{b}) = \emptyset$). In particular, $q_{|\,\mathrm{Sing}(q)}\colon \mathrm{Sing}(q) \to Y$ is an isomorphism (it is quasi–finite of degree 1); thus there is an invertible $\mathcal{O}_Y$–sheaf $\mathcal{L}$ such that the section $Y \xrightarrow{\sim} \mathrm{Sing}(q) \subseteq \mathbb{P}$ corresponds to the exact sequence $0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{L} \to 0$. Let $f\colon \mathbb{P} \setminus \mathrm{Sing}(q) \to \mathbb{P}(\mathcal{E}_0)$ be the induced dominant map (which is, fibrewise, the projection from the point $\mathrm{Sing}(Q_y)$ onto the line $\mathbb{P}(\mathcal{E}_0)_y$) and define $X_0 := f(X)$ (which is, fibrewise, $Q_y \cap \mathbb{P}(\mathcal{E}_0)_y \subseteq \mathbb{P}_y$). Let $\alpha := f_{|X}\colon X \to X_0$. The projection $\mathbb{P}(\mathcal{E}_0) \to Y$ induces a double cover $\beta\colon X_0 \to Y$ such that $\varrho = \beta \circ \alpha$. The ramification points of β correspond to conics Q_y of rank 1; hence β is necessarily étale. Then we assume that

($\heartsuit$) ϱ does not factor as $\beta \circ \alpha$ with β a non–trivial étale double cover of Y.

Condition ($\heartsuit$) is obviously satisfied if Y is simply connected.

If ($\heartsuit$) is satisfied one gets that X_0 is trivial, and this allows us to find a decomposition $Q = \mathbb{P}_1 \cup \mathbb{P}_2$ into two projective subbundles $\mathbb{P}_i \subseteq \mathbb{P}$ of rank 1. But, if this is the case, $X = X_1 \cup X_2$, with $X_i := X \cap \mathbb{P}_i \neq \emptyset$, since, generically, the fibre $X_y := \varrho^{-1}(y)$ consists of four points in general position in $\pi^{-1}(y) \cong \mathbb{P}^2_k$.

Proposition 4.9. *Let $\varrho\colon X \to Y$ be a cover of degree 4 with X and Y smooth and integral and $R_{planar}(\varrho) = \emptyset$. If Y satisfies ($\heartsuit$) then $\Delta(X)$ is connected.* □

Corollary 4.10. *If ϱ, X and Y are as above, then $h^0\big(Y, \check{\mathcal{F}}\big) = 0$.*

Proof. The number of connected components of $\Delta(X)$ is

$$h^0\big(\Delta(X), \mathcal{O}_{\Delta(X)}\big) = h^0\big(Y, \mathcal{O}_Y\big) + h^0\big(Y, \check{\mathcal{F}}\big).$$

□

Corollary 4.11. *If ϱ, X and Y are as above and B_ϱ is reduced (resp. has at most ordinary cuspidal double points as singularities), then $\Delta(X)$ is integral (resp. smooth).*

Proof. We already know that $\Delta(X)$ is reduced. If $\Delta(X) \subseteq \mathbb{P}(\mathcal{F})$ were reducible then we would have $\dim\big(\mathrm{Sing}(\Delta(X))\big) \geq \dim(X) - 1$ (since $\Delta(X)$ is connected). On the other hand,

$$\begin{aligned}\mathrm{Sing}(\Delta(X)) &\subseteq \Delta(\varrho)^{-1}\{y \in Y |\ S(X_y) = [(1,1),1],\ [(2,1)]\} \\ &\subseteq \Delta(\varrho)^{-1}(\mathrm{Sing}(B_{\Delta(\varrho)})) = \Delta(\varrho)^{-1}(\mathrm{Sing}(B_\varrho)).\end{aligned}$$

Hence $\dim\big(\mathrm{Sing}(\Delta(X))\big) \leq \dim(\mathrm{Sing}(B_\varrho)) \leq \dim(X) - 2$.

If B_ϱ has at most ordinary cuspidal double points as singularities, then the smoothness of $\Delta(X)$ follows from section 3. □

Both Proposition 4.9 and its Corollaries 4.10 and 4.11 are sharp, as the following easy examples show.

Example 4.12. Consider Example 4.6. As we saw, $\Delta(X)$ is in any case reducible. On the other hand, using the expression of b given by (3.5), one finds that the branch locus B_ϱ has global equation

$$b = 4\beta^3\gamma^2 + \beta^2\delta^2 = (4\beta\gamma^2 + \delta^2)\beta^2,$$

which is not reduced.

Example 4.13. Consider an elliptic curve Y over $\mathbb{C}$, and let $\beta\colon X_0 \to Y$ be a double étale cover with X_0 smooth and connected. Then there is an invertible sheaf $\mathcal{L}$ such that $\mathcal{L}^2 \cong \mathcal{O}_Y$ and $\beta_*\mathcal{O}_{X_0} \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}$. Now let $\mathcal{M}$ be an invertible and globally generated $\mathcal{O}_Y$–sheaf and let $\alpha\colon X \to X_0$ be any smooth and connected double cover with $\alpha_*\mathcal{O}_X \cong \mathcal{O}_{X_0} \oplus \beta^*\mathcal{M}^{-1}$. The morphism $\varrho := \beta \circ \alpha\colon X \to Y$ is a Gorenstein cover of degree 4 whose Tschirnahausen module is dual to $\mathcal{E} := \mathcal{L} \oplus \mathcal{M} \oplus \mathcal{L} \otimes \mathcal{M}$. Moreover, $\omega_{X|k} \cong \omega_{X|X_0} \otimes \omega_{X_0|Y} \cong \varrho^*\big(\mathcal{L} \otimes \mathcal{M}\big)$. It is not difficult to check that $\mathcal{F} \cong \mathcal{L}^2 \oplus \mathcal{M}^2 \cong \mathcal{O}_Y \oplus \mathcal{M}^2$. Hence $h^0\big(Y, \check{\mathcal{F}}\big) = h^0\big(Y, \mathcal{O}_Y\big) + h^0\big(Y, \mathcal{M}^{-2}\big) = 1$, so that $\Delta(X)$ is not connected. Note that, since X is smooth, $\Delta(\varrho)\colon \Delta(X) \to Y$ is actually a cover of degree 3.

5. Singularities of the discriminant

In section 4 we gave a first rough description of $\mathrm{Sing}(\Delta(X))$. In this section we want to study the nature of the singularities of the discriminant of a cover $\varrho\colon X \to Y$ of degree 4 under the hypothesis of genericity. We will assume that X is smooth and Y is smooth and integral.

To that end we inspect the construction of the discriminant as the degeneration divisor of the conic bundle $p\colon Bl_X\mathbb{P} \to \overline{\mathbb{P}}$ used in the previous section. For each $x \in \mathbb{P} \setminus X$ let $y := \pi(x)$, and let C_x be the unique conic contained in $\mathbb{P}_y$ and containing both x and X_y. Then $C_x \in \overline{\mathbb{P}}_y$. We have a commutative diagram

$$\begin{array}{ccccc} E & \hookrightarrow & Bl_X\mathbb{P} & \xrightarrow{p} & \overline{\mathbb{P}} \\ \big\downarrow & & \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\overline{\pi}} \\ X & \hookrightarrow & \mathbb{P} & \xrightarrow{\pi} & Y \end{array} \tag{5.1}$$

where φ is the natural projection. Since the fibres of p are conics, it follows that p is a Gorenstein morphism (of relative dimension 1). Hence, so is $\overline{\pi} \circ p$, and the relative dualizing sheaf $\omega_{Bl_X\mathbb{P}|Y}$ is invertible. Moreover, by semicontinuity, $\mathcal{H}' := p_*\omega^{-1}_{Bl_X\mathbb{P}|Y}$ is a locally free $\mathcal{O}_{\overline{\mathbb{P}}}$–sheaf of rank 2 and the natural epimorphism $p^*\mathcal{H}' \twoheadrightarrow \omega_{Bl_X\mathbb{P}|Y}$ induces an embedding $Bl_X\mathbb{P} \hookrightarrow \widetilde{\mathbb{P}} := \mathbb{P}(\mathcal{H}')$ factoring p through the natural projection $\widetilde{p}\colon \widetilde{\mathbb{P}} \to \overline{\mathbb{P}}$. In particular, $p\colon Bl_X\mathbb{P} \to \overline{\mathbb{P}}$ is an embedded conic in the sense of [Sa], 1.5.

Since p is induced by the relative conics through X, it follows that $p^*\mathcal{O}_{\overline{\mathbb{P}}}(1) \cong \varphi^*\mathcal{O}_{\mathbb{P}}(2) \otimes \mathcal{O}_{Bl_X\mathbb{P}}(-E)$. In particular,

$$\omega_{Bl_X\mathbb{P}|Y} \cong \varphi^*\omega_{\mathbb{P}|Y} \otimes \mathcal{O}_{Bl_X\mathbb{P}}(E) \cong p^*\mathcal{O}_{\overline{\mathbb{P}}}(-1) \otimes \varphi^*\mathcal{O}_{\mathbb{P}}(-1) \otimes \varphi^*\pi^*\det\mathcal{E}.$$

Then $\mathcal{H}' \cong \mathcal{O}_{\overline{\mathbb{P}}}(1) \otimes p_*\varphi^*\big(\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\det\mathcal{E}^{-1}\big)$. Since π, $\overline{\pi}$ and p are flat, then $p_*\varphi^* \cong \overline{\pi}^*\pi_*$ as functors (see [Ha], proposition III 9.3); hence $\mathcal{H}' \cong \mathcal{H} := \mathcal{O}_{\overline{\mathbb{P}}}(1) \otimes \overline{\pi}^*\big(\mathcal{E} \otimes \det\mathcal{E}^{-1}\big)$. $Bl_X\mathbb{P}$ has an $\mathcal{O}_{\widetilde{\mathbb{P}}}$–resolution of the form

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}}(-2) \otimes \widetilde{p}^*\mathcal{M}^{-1} \to \mathcal{O}_{\widetilde{\mathbb{P}}} \to \mathcal{O}_{Bl_X\mathbb{P}} \to 0,$$

where $\mathcal{M} \cong \mathcal{O}_{\overline{\mathbb{P}}}(-1) \otimes \overline{\pi}^*\det\mathcal{F}$ (see [Sa], 1.5 and 1.6). Note that $\mathcal{S}^2\,\mathcal{H} \otimes \mathcal{M} \cong \overline{\pi}^*\big(\mathcal{S}^2\,\mathcal{E} \otimes \det\mathcal{E}^{-1}\big) \otimes \mathcal{O}_{\overline{\mathbb{P}}}(1)$; in particular, $\overline{\pi}_*\big(\mathcal{S}^2\,\mathcal{H} \otimes \mathcal{M}\big) \cong \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}$, and there is a natural isomorphism

$$\Phi\colon H^0\big(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big) \xrightarrow{\sim} H^0\big(\overline{\mathbb{P}}, \mathcal{S}^2\,\mathcal{H} \otimes \mathcal{M}\big) \cong \mathrm{Hom}^{sym}_{\mathcal{O}_{\overline{\mathbb{P}}}}(\check{\mathcal{H}}, \mathcal{H} \otimes \mathcal{M}).$$

The symmetric map $\vartheta = \Phi(\eta)\colon \check{\mathcal{H}} \to \mathcal{H} \otimes \mathcal{M}$ yields an exact sequence

$$0 \to \check{\mathcal{H}} \xrightarrow{\vartheta} \mathcal{H} \otimes \mathcal{M} \to \mathcal{P} \to 0, \tag{5.2}$$

where $\mathcal{P}$ is supported on $\Delta(X)$. Notice that $\Delta(X) = D_2(\vartheta)$ (by construction) and $\mathrm{Sing}(\Delta(X)) = D_1(\vartheta)$ (Proposition 4.5 iii)).

With this in mind we can use the results of [Ba] to describe the singularities of $\Delta(X)$.

Proposition 5.3. *Let $\mathcal{E}$, $\mathcal{F}$ be locally free sheaves of ranks 3 and 2 respectively over an integral and smooth scheme Y of dimension 2.*

If $\mathcal{S}^2\,\mathcal{E} \otimes \check{\mathcal{F}}$ is globally generated, then the discriminant $\Delta(X)$ of each general cover $\varrho\colon X \to Y$ of degree 4 with invariants $\mathcal{E}$ and $\mathcal{F}$ is smooth in codimension 1 and has at most nodes as singularities.

Proof. The natural map $\overline{\pi}^*\mathcal{F} \twoheadrightarrow \mathcal{O}_{\overline{\mathbb{P}}}(1)$ induces an epimorphism $\overline{\pi}^*\big(\mathcal{S}^2\,\mathcal{E} \otimes \check{\mathcal{F}}\big) \twoheadrightarrow \mathcal{S}^2\,\mathcal{H} \otimes \mathcal{M}$. If $\mathcal{S}^2\,\mathcal{E} \otimes \check{\mathcal{F}}$ is globally generated, then the same is true for $\mathcal{S}^2\,\mathcal{H} \otimes \mathcal{M}$. Thus the proposition follows from proposition 1 of [Ba]. □

Proposition 5.4. *Let $\varrho\colon X \to Y$ be a cover of degree 4 with X smooth, Y smooth, integral and projective, and invariants $\mathcal{E}$, $\mathcal{F}$. If $\Delta(X)$ is smooth in codimension 1, then $\mathrm{Sing}(\Delta(X))$ represents the class of*

$$4\overline{\pi}^*\left(c_2(\mathcal{E}) - c_2(\mathcal{F})\right) \cdot \xi - 4\overline{\pi}^* c_3(\mathcal{E}), \tag{5.4.1}$$

$\xi \in \mathrm{Pic}(\overline{\mathbb{P}})$ being the tautological class. In particular, if $\dim(Y) = 2$ and $\Delta(X)$ has at most nodes as singularities, then

$$\#\,\mathrm{Sing}(\Delta(X)) = 4\left(c_2(\mathcal{E}) - c_2(\mathcal{F})\right). \tag{5.4.2}$$

Proof. To prove the statement we consider formula 9 of [Ba]. In this case $r = 3$, $c_i = c_i(\mathcal{H})$, $\lambda = c_1(\mathcal{M})$, where $\mathcal{M} \cong \mathcal{O}_{\overline{\pi}}(-1) \otimes \overline{\pi}^* \det \mathcal{E}$ and $\mathcal{H} = \overline{\pi}^*\mathcal{E} \otimes \mathcal{M}^{-1}$. A direct substitution in Barth's formula then proves that $\mathrm{Sing}(\Delta(X))$ represents

$$4\left(\xi^3 - \overline{\pi}^* c_1(\mathcal{E}) \cdot \xi^2 + \overline{\pi}^* c_2(\mathcal{E}) \cdot \xi - \overline{\pi}^* c_3(\mathcal{E})\right).$$

Since $\xi^3 - \overline{\pi}^* c_1(\mathcal{F}) \cdot \xi^2 + \overline{\pi}^* c_2(\mathcal{F}) \cdot \xi = 0$ and $c_1(\mathcal{E}) = c_1(\mathcal{F})$, we get formula (5.4.1). Formula (5.4.2) follows by projecting via $\overline{\pi}_*$ and the fact that the singularities of $\Delta(X)$ are nodes. □

Example 5.5. Let X be an Enriques surface not containing nodal curves. There always exists a cover $\varrho\colon X \to \mathbb{P}^2_k$ of degree 4 whose invariants are $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2_k}(2)^{\oplus 3}$ and the kernel $\mathcal{F}$ of an epimorphism $\mathcal{O}_{\mathbb{P}^2_k}(4)^{\oplus 3} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2_k}(6)$ (see section 6 of [C–E] and [Ve]). In particular, $c_2(\mathcal{E}) = 12 = c_2(\mathcal{F})$. Since in this case $\mathcal{S}^2\,\mathcal{E} \otimes \check{\mathcal{F}}$ is globally generated, for each such general ϱ the discriminant Δ is smooth.

Indeed, A. Verra proved in [Ve] that the branch locus of ϱ is a reduced curve of degree 12 having ordinary cuspidal points as singularities; thus $R_{even}(\varrho) = \emptyset$ by Corollary 4.11.

6. The trigonal construction

In this section we want to investigate the connection between the discriminant of a cover of degree 4 and a generalization of the well known trigonal construction due to S. Recillas (see [Re]).

We maintain the notations of sections 4 and 5. The first step is to study the sheaf $\mathcal{P}$ fitting into the sequence (5.2).

Definition 6.1. Let Δ be a scheme defined over a field of characteristic $p \neq 2$, $\mathcal{L}$ an invertible $\mathcal{O}_\Delta$–sheaf and $\mathcal{P}$ a coherent reflexive $\mathcal{O}_\Delta$–sheaf. We say that $\mathcal{P}$ is an $\mathcal{L}$–quadratic sheaf if there exists a symmetric isomorphism

$$\sigma\colon \mathcal{P} \xrightarrow{\sim} \check{\mathcal{P}} \otimes \mathcal{L} \cong \mathcal{H}om_{\mathcal{O}_\Delta}\big(\mathcal{P}, \mathcal{L}\big).$$

Lemma 6.2. *If $\Delta(X)$ is normal, then the $\mathcal{O}_{\Delta(X)}$–sheaf $\mathcal{P}$ fitting in the sequence 5.2 above is an $\omega^2_{\Delta(X)|Y}$–quadratic sheaf.*

$\mathcal{P}$ is not invertible exactly on $\mathrm{Sing}(\Delta(X))$.

Proof. Taking the tensor product of sequence 5.2 with itself, we obtain

(6.2.1)

$$\big(\check{\mathcal{H}} \otimes \mathcal{H} \otimes \mathcal{M}\big) \oplus \big(\mathcal{H} \otimes \mathcal{M} \otimes \check{\mathcal{H}}\big) \xrightarrow{\delta_1} \big(\mathcal{H} \otimes \mathcal{M}\big) \otimes \big(\mathcal{H} \otimes \mathcal{M}\big) \to \mathcal{P} \otimes \mathcal{P} \to 0,$$

where $\delta_1(\alpha \otimes a, b \otimes \beta) = \vartheta(\alpha) \otimes a + b \otimes \vartheta(\beta)$ and which is exact (use the spectral sequence of a double complex). On the other hand, let $\mathfrak{S}_2 := \{1, i\}$. The map $T_0 := 1 + i$ acts on $\big(\mathcal{H} \otimes \mathcal{M}\big) \otimes \big(\mathcal{H} \otimes \mathcal{M}\big)$ sending $a \otimes b$ to $a \otimes b + b \otimes a$. In particular, T_0 extends to an endomorphism T of (6.2.1) whose image is a direct summand, which is isomorphic to

$$\check{\mathcal{H}} \otimes \mathcal{H} \otimes \mathcal{M} \xrightarrow{d_1} \mathcal{S}^2\big(\mathcal{H} \otimes \mathcal{M}\big) \to \mathcal{S}^2\,\mathcal{P} \to 0 \tag{6.2.2}$$

via the canonical projection. It follows that 6.2.2 is exact.

The adjoint of ϑ maps $\mathcal{H} \otimes \mathcal{M}$ to $\check{\mathcal{H}}(\Delta(X))$; hence it induces a morphism $\phi_1\colon \mathcal{S}^2\big(\mathcal{H} \otimes \mathcal{M}\big) \to \mathcal{M}(\Delta(X))$. Denoting by $\phi_2\colon \check{\mathcal{H}} \otimes \mathcal{H} \otimes \mathcal{M}$ the usual contraction, we then get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\check{\mathcal{H}} \otimes \mathcal{H} \otimes \mathcal{M} & \xrightarrow{d_1} & \mathcal{S}^2\big(\mathcal{H} \otimes \mathcal{M}\big) & \to & \mathcal{S}^2\,\mathcal{P} & \to & 0 \\
\downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & & & \\
\mathcal{M} & \to & \mathcal{M}(\Delta(X)) & \to & \mathcal{M}(\Delta(X))_{|\Delta(X)} & \to & 0,
\end{array}$$

giving rise to a map $\phi_0\colon\ \mathcal{S}^2\,\mathcal{P} \to \mathcal{M}(\Delta(X))_{|\Delta(X)} \cong \omega^2_{\Delta(X)|Y}$, hence to a morphism $\sigma\colon \mathcal{P} \to \check{\mathcal{P}} \otimes \omega^2_{\Delta(X)|Y}$. Since $\mathrm{Sing}(\Delta(X)) = D_1(\vartheta)$, we see that ϕ_1 is surjective outside $\mathrm{Sing}(\Delta(X))$ and the same is true for ϕ_0. It follows that σ is an isomorphism outside $\mathrm{Sing}(\Delta(X))$, since $\mathcal{P}$ is invertible there (see [Ba], Lemma 5).

Taking the dual of sequence (5.2) twisted by $\mathcal{M}$, one gets

$$0 \to \check{\mathcal{H}} \to \mathcal{H} \otimes \mathcal{M} \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}\big(\mathcal{P}, \mathcal{M}\big) \to 0. \tag{6.2.3}$$

The short exact sequence $0 \to \mathcal{M} \to \mathcal{M}(\Delta(X)) \to \mathcal{M}(\Delta(X))_{|\Delta(X)} \to 0$ yields an isomorphism $\partial\colon \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}\big(\mathcal{P}, \mathcal{M}\big) \xrightarrow{\sim} \check{\mathcal{P}} \otimes \omega^2_{\Delta(X)|Y}$. In particular, locally on Y, σ lifts to a chain map between sequences (5.2) and (6.2.3), and any two such maps are homotopic. For degree reasons any homotopy must be zero, so we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \check{\mathcal{H}} & \xrightarrow{\vartheta} & \mathcal{H} \otimes \mathcal{M} & \to & \mathcal{P} & \to & 0 \\
 & & \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle \sigma} & & \\
0 & \to & \check{\mathcal{H}} & \xrightarrow{\check{\vartheta}} & \mathcal{H} \otimes \mathcal{M} & \to & \check{\mathcal{P}} \otimes \omega^2_{\Delta(X)|Y} & \to & 0
\end{array} \tag{6.2.4}$$

Sequences (5.2) and (6.2.3) are minimal, so s_1 and s_2 are isomorphisms outside $\mathrm{Sing}(\Delta(X))$. The normality of $\Delta(X)$ then implies that they are both isomorphisms everywhere, so the same must be true for σ.

$\mathcal{P}$ is reflexive. Indeed, $\mathcal{P}$ is torsion–free, so the natural map $\mu\colon \mathcal{P} \to \mathcal{P}^{\check{}\check{}}$ is injective. On the other hand, $\mathcal{P}$ and $\mathcal{P}^{\check{}\check{}}$ are isomorphic via $(\sigma^t)^{-1}\circ\sigma$. Thus $\mathcal{P}$ and $\mathcal{P}^{\check{}\check{}}$ have the same Hilbert function, and it follows that $\operatorname{coker}(\mu)=0$.

Since $\mathcal{P}$ is reflexive and σ is induced by ϕ_0 it follows that σ is symmetric, i.e. $\mathcal{P}$ is an $\omega^2_{\Delta(X)|Y}$–quadratic sheaf.

Finally, $\mathcal{P}$ is invertible outside $\operatorname{Sing}(\Delta(X))$. On the other hand, if $x \in \operatorname{Sing}(\Delta(X))$ then $y := \pi(x)$ is even or planar (Proposition 4.5 iii)). In the first case $X_y = C\cap D$, where $C = V_+(u^2+v^2)$ and $D = V_+(w^2)$; thus $t^2 s$ is an equation for $\Delta(X)_y \subseteq \overline{\mathbb{P}}_y$ and ϑ_y is defined by the matrix

$$\begin{pmatrix} t & 0 & 0\\ 0 & t & 0\\ 0 & 0 & s\end{pmatrix}.$$

Note that $D \in \Delta(X)_y$ corresponds to $s=1$. We conclude that

$$\dim_{k(D)}\big(\mathcal{P}\otimes k(D)\big)=2,$$

and so $\mathcal{P}$ cannot be invertible at D. The cases $S(X_y) = [(2,1)],\ [1,1;;1]$ can be treated analogously. $\square$

Definition 6.3. Let Y be integral, defined over a field of characteristic $p\neq 2$, and let $\mathcal{E}$ and $\mathcal{F}$ be locally free $\mathcal{O}_Y$–sheaves of rank 3 and 2 respectively.

Let $\mathcal{T}^4_{Y,\mathcal{E},\mathcal{F}}$ denote the set of Gorenstein covers $X \xrightarrow{\varrho} Y$ of degree 4 with invariants $\mathcal{E}$, $\mathcal{F}$ and such that X is smooth and $\operatorname{codim}_X\big(R_{even}(\varrho)\big)\geq 2$, $R_{planar}(\varrho)=\emptyset$.

Let $\mathcal{R}^3_{Y,\mathcal{E},\mathcal{F}}$ denote the set of pairs $(\mathcal{P},\Delta \xrightarrow{\tau} Y)$ such that τ is a Gorenstein cover of degree 3 with invariant $\mathcal{F}$, Δ is normal with at most pseudo–nodes as singularities, and $\mathcal{P}$ is an $\omega^2_{\Delta|Y}$–quadratic sheaf which is invertible exactly outside $\operatorname{Sing}(\Delta)$ and such that $\tau_*\mathcal{P}\cong\mathcal{E}$.

Up to now we have shown how to associate to each $X \xrightarrow{\varrho} Y \in \mathcal{T}^4_{Y,\mathcal{E},\mathcal{F}}$ an element $Trig(X \xrightarrow{\varrho} Y)\in\mathcal{R}^3_{Y,\mathcal{E},\mathcal{F}}$.

Definition 6.4. The map

$$Trig\colon \mathcal{T}^4_{Y,\mathcal{E},\mathcal{F}} \to \mathcal{R}^3_{Y,\mathcal{E},\mathcal{F}}.$$

defined above is called the *"trigonal construction"*.

Theorem 6.5. *The map $Trig$ is bijective.*

Proof. We only need to show how to recover a Gorenstein cover of degree 4 from a fixed pair $(\mathcal{P},\Delta \xrightarrow{\tau} Y)$.

First of all, assume that $Y=\operatorname{spec}(k)$ is a point, i.e. $\overline{\mathbb{P}}\cong\mathbb{P}^1_k$. Then $H^0\big(Y,\mathcal{P}\big)\cong k^{\oplus 3}$, $\mathcal{P}$ is 0–regular since $\dim(\Delta)=0$ and it is supported on Δ. In particular the natural maps $H^0\big(Y,\mathcal{P}\big)\otimes H^0\big(Y,\mathcal{O}_{\mathbb{P}^1_k}(n)\big)\to H^0\big(Y,\mathcal{P}(n)\big)$ are surjective for each $n\geq 0$. Thus there exists an epimorphism $\varphi\colon \mathcal{O}^{\oplus 3}_{\mathbb{P}^1_k}\twoheadrightarrow\mathcal{P}$ whose kernel is free, say $\ker(\varphi)\cong\bigoplus_{i=1}^3\mathcal{O}_{\mathbb{P}^1_k}(-\alpha_i)$, as follows from the Auslander–Buchsbaum formula and the decomposition theorem of Grothendieck. It follows that an exact sequence of the form

$$0\to\bigoplus_{i=1}^{3}\mathcal{O}_{\mathbb{P}^1_k}(-\alpha_i)\to\mathcal{O}^{\oplus 3}_{\mathbb{P}^1_k}\xrightarrow{\varphi}\mathcal{P}\to 0 \tag{6.5.1}$$

is defined. Since the map $H^0(\varphi)$ is an isomorphism, taking the cohomologies of (6.5.1) one obtains $\alpha_i = 1$ for $i = 1, 2, 3$.

Now consider an arbitrary scheme Y and let $\mathcal{M}$ and $\mathcal{H}$ be as in section 5. One obtains the exact sequence

$$0 \to \mathcal{K} \xrightarrow{\theta} \mathcal{H} \otimes \mathcal{M} \to \mathcal{P} \to 0, \tag{6.5.2}$$

whose restriction to $\overline{\mathbb{P}}_y$ coincides with (6.5.1). Since $h^1(\overline{\mathbb{P}}_y, \mathcal{K}_{|\overline{\mathbb{P}}_y}) = 0$ and $h^0(\overline{\mathbb{P}}_y, \mathcal{K}_{|\overline{\mathbb{P}}_y}) = 3$, thus $R^1\overline{\pi}_*\mathcal{K}(1) = 0$ and $\mathcal{N} := \overline{\pi}_*\mathcal{K}(1)$ is locally free of rank 3. Moreover, the natural map $\overline{\pi}^*\mathcal{N} \to \mathcal{K}(1)$ is an isomorphism since this is true fibrewise.

The symmetric map σ induces $\mathcal{S}^2\,\mathcal{P} \to \omega^2_{\Delta|Y} \cong \mathcal{M}(\Delta)_{|\Delta}$, hence a diagram

$$\begin{array}{ccccccccccc} 0 & \to & \Lambda^2\mathcal{K} & \xrightarrow{d_2} & \mathcal{K}\otimes\mathcal{H}\otimes\mathcal{M} & \xrightarrow{d_1} & \mathcal{S}^2\left(\mathcal{H}\otimes\mathcal{M}\right) & \to & \mathcal{S}^2\,\mathcal{P} & \to & 0 \\ & & & & & & & & \downarrow{\scriptstyle \phi_0} & & \\ & & 0 & \to & \mathcal{M} & \to & \mathcal{M}(\Delta) & \to & \mathcal{M}(\Delta)_{|\Delta} & \to & 0. \end{array}$$

Locally on the base the map ϕ_0 can be lifted to a chain map, and any two such maps are homotopic. On the other hand, since $\mathcal{K} \cong \overline{\pi}^*\mathcal{N}(-1)$, for degree reasons each homotopy must be zero, so we obtain a well defined chain map and a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & \Lambda^2\mathcal{K} & \xrightarrow{d_2} & \mathcal{K}\otimes\mathcal{H}\otimes\mathcal{M} & \xrightarrow{d_1} & \mathcal{S}^2\left(\mathcal{H}\otimes\mathcal{M}\right) & \to & \mathcal{S}^2\,\mathcal{P} & \to & 0 \\ & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \\ & & 0 & \to & \mathcal{M} & \to & \mathcal{M}(\Delta) & \to & \mathcal{M}(\Delta)_{|\Delta} & \to & 0. \end{array}$$

The map ϕ_2 induces a morphism $s_1\colon \mathcal{H} \to \check{\mathcal{K}}$, defined locally by $\langle s_1(\alpha), \beta\rangle := \phi_2(\alpha \otimes \beta)$. Let s_2 be the transpose of s_1: we claim that $s_1 \circ \theta = \check{\theta} \circ s_2$. Indeed, we have locally

$$0 = \phi_3(\alpha \wedge \beta) = \phi_2 \circ d_2(\alpha \wedge \beta) = \phi_2(\alpha \otimes \theta(\beta) - \beta \otimes \theta(\alpha)).$$

Thus

$$\langle s_1 \circ \theta(\alpha), \beta\rangle = \phi_2(\beta \otimes \theta(\alpha)) = \phi_2(\alpha \otimes \theta(\beta)) = \langle s_2(\alpha), \theta(\beta)\rangle = \langle \check{\theta} \circ s_2(\alpha), \beta\rangle.$$

It follows that there exists a chain map

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \xrightarrow{\theta} & \mathcal{H}\otimes\mathcal{M} & \to & \mathcal{P} & \to & 0 \\ & & \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle \sigma} & & \\ 0 & \to & \check{\mathcal{H}} & \xrightarrow{\check{\theta}} & \check{\mathcal{K}}\otimes\mathcal{M} & \to & \check{\mathcal{P}}\otimes\omega^2_{\Delta|Y} & \to & 0 \end{array}$$

extending the isomorphism σ. By construction each row of the above diagram is minimal; thus the maps s_1 and s_2 are isomorphisms and we get the exact sequence

$$0 \to \check{\mathcal{H}} \xrightarrow{\vartheta} \mathcal{H} \otimes \mathcal{M} \to \mathcal{P} \to 0,$$

where $\vartheta := \theta \circ s_2^{-1}$. Notice that $\vartheta = \check{\vartheta}$. Hence we obtain $\eta \in H^0(Y, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E})$ via the isomorphism Φ^{-1} defined in section 5.

We claim that η has the right codimension at each point $y \in Y$. Indeed, η defines a subscheme $X := D_0(\eta) \subseteq \mathbb{P}$ which is fibrewise over $y \in Y$ the base locus of a pencil of conics $\mathfrak{b}_y$ whose discriminant is Δ_y. Since τ is a cover we have $\Delta_y \neq \mathbb{P}^1_{k(y)}$.

Hence $\mathfrak{b}_y$ has no fixed components, and $\varrho := \pi_{|X} \colon X \to Y$ is a Gorenstein cover of degree 4 without planar points.

It remains to prove the smoothness of X. Since $\mathcal{P}$ is not invertible exactly on $\mathrm{Sing}(\Delta)$, it follows that $\varrho(\mathrm{Sing}(X)) \subseteq \varrho(R_{even}(\varrho)) = \tau(\mathrm{Sing}(\Delta))$. Let $y \in \varrho(R_{even}(\varrho))$ be such that $S(X_y)$ is either $[(2,1)]$ or $[(1,1),1]$. Then $X_y = C \cap D$ where $C = V_+(u^2)$ and $D = V_+(v^2+2u)$ (so $\mathrm{Supp}(X_y) = (0,0)$) in the first case or $C = V_+(u^2)$ and $D = V_+(v^2-1)$ (so $\mathrm{Supp}(X_y) = \{(0,\pm 1)\}$) in the second. Locally on Y the subscheme $X \subseteq \mathbb{P}$ is given by $\mathrm{spec}(R)$, where R is as in (3.4), and either $\alpha(y) = \beta(y) = \delta(y) = 0$ and $\gamma(y) = 1$ in the first case, or $\alpha(y) = \beta(y) = \gamma(y) = 0$ and $\delta(y) = -1$ in the second (see (2.2) and the description above). Then Δ is given by (4.2) locally on Y. Moreover, $C \in \mathrm{Sing}(\Delta)$ corresponds to $s = 0$. Let $\{y_1, \dots, y_n\}$ be a system of local regular parameters around y, and assume that X is singular over y. If $S(X_y) = [(2,1)]$, one can easily check that the jacobian matrix of X at $u = v = 0$ is

$$\begin{pmatrix} 0 & 0 & \frac{\partial\beta}{\partial y_i} \\ 2 & 0 & \frac{\partial\delta}{\partial y_i} \end{pmatrix}_{i=1,\dots,n},$$

whence $\left(\frac{\partial\beta}{\partial y_i}\right)_{|y} = 0$. If $S(X_y) = [(1,1),1]$, the jacobian matrix at $u = 0$, $v = \pm 1$ is

$$\begin{pmatrix} 0 & 0 & \pm 2\frac{\partial\alpha}{\partial y_i} + \frac{\partial\beta}{\partial y_i} \\ 0 & \pm 2 & \frac{\partial\delta}{\partial y_i} \end{pmatrix}_{i=1,\dots,n},$$

whence $\left(\frac{\partial\alpha}{\partial y_i}\right)_{|y} = \pm 2i\left(\frac{\partial\beta}{\partial y_i}\right)_{|y} = 0$. In both the cases the tangent cone at $C \in \Delta$ would have rank at most 1. □

Remark 6.6. It is not difficoult to verify that giving a $\omega^2_{\Delta|Y}$–quadratic sheaf $\mathcal{P}$ on Δ is equivalent to giving a finite morphism $\Delta' \xrightarrow{\tau'} \Delta$ of degree 2 branched exactly at $\mathrm{Sing}(\Delta)$. Indeed, we define the $\mathcal{O}_\Delta$–algebra $\mathcal{A} := \mathcal{O}_\Delta \oplus \left(\mathcal{P} \otimes \omega^{-1}_{\Delta|Y}\right)$ with the multiplication induced by σ and consider the canonical projection $\Delta' := \mathbf{Spec}(\mathcal{A})$ onto Δ.

We give the following classical example.

Example 6.7. Let $X := \mathbb{P}^1_k \times_k \mathbb{P}^1_k$. The linear system $\mathcal{O}_X(1,2)$ induces an embedding $i \colon X \hookrightarrow \mathbb{P}^5_k$. Projecting $i(X)$ onto a non–intersecting plane α, we finally get a cover $\varrho \colon X \to \mathbb{P}^2_k$ of degree 4. The invariants of ϱ are $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2_k}(1)^{\oplus 3}$ and $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2_k}(1) \oplus \mathcal{O}_{\mathbb{P}^2_k}(2)$. It follows from Proposition 5.3 and formula (5.4.2) that for a general projection the discriminant Δ has exactly 4 nodes as singularities. If $\overline{\mathbb{P}} \cong Bl_r\mathbb{P}^3_k \xrightarrow{\varphi} \mathbb{P}^3_k$ is the blow-up of $\mathbb{P}^3_k$ along a line r, then $\varphi(\Delta) \subseteq \mathbb{P}^3_k$ is a cubic and $\varphi_{|\Delta}$ is an isomorphism. We conclude that $\varphi(\Delta)$ is a Cayley cubic. On the other hand, we also have a double cover $\Delta' \xrightarrow{\tau'} \Delta$ branched along $\mathrm{Sing}(\Delta)$. One has $\tau_*\tau'_*\mathcal{O}_{\Delta'} \cong \mathcal{O}_{\mathbb{P}^2_k} \oplus \mathcal{O}_{\mathbb{P}^2_k}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^2_k}(-2)$. Δ' is a Del Pezzo sextic in $\mathbb{P}^6_k$, and $\tau\tau'$ is the projection from a space $\beta \subseteq \mathbb{P}^6_k$ of dimension 3.

It is a classical result that all the Cayley cubic surfaces arise in this way.

7. Covers of $\mathbb{P}^n_{\mathbb{C}}$.

We defined $R_{planar}(\varrho)$ in section 4 for any cover $\varrho \colon X \to Y$ of degree 4 with both X and Y smooth and integral. In this section we want to restrict ourselves to the case $Y \cong \mathbb{P}^n_{\mathbb{C}}$, and we will deal with the behaviour of ϱ with respect to $R_{planar}(\varrho)$ when $n \geq 5$. In this case there always exist points $y \in Y$ of total ramification (see [G–L]).

Proposition 7.1. *Let X be integral and smooth, $n \geq 5$, and let $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ be a cover of degree 4. Then*

i) there is $k \geq 2$ such that $\omega_{X|\mathbb{P}^n_{\mathbb{C}}} \cong \varrho^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k)$;

ii) for each line $r \subseteq \mathbb{P}^n_{\mathbb{C}}$ there are $k_2 \leq k_1 < k$, depending on r, such that $k_1 + k_2 = k$, $\mathcal{E}_{|r} \cong \mathcal{O}_r(k) \oplus \mathcal{O}_r(k_1) \oplus \mathcal{O}_r(k_2)$ and $\mathcal{F}_{|r} \cong \mathcal{O}_r(2k_1) \oplus \mathcal{O}_r(2k_2)$;

iii) $c_1(\mathcal{F}) = c_1(\mathcal{E}) = 2k$, $c_2(\mathcal{F}) = 4c_2(\mathcal{E}) - k^2$.

Proof. If $n \geq 5$ then $\omega_{X|\mathbb{P}^n_{\mathbb{C}}} \cong \varrho^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k)$ for some $k \neq 0$ (see [La], proposition 3.1). Moreover, the isomorphisms $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-k) \oplus \mathcal{E}(-k) \cong \big(\varrho_*\omega_{X|\mathbb{P}^n_{\mathbb{C}}}\big)(-k) \cong \varrho_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}} \oplus \check{\mathcal{E}}$ give rise to a factorization of the identity on $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$ as $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}} \xrightarrow{i} \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-k) \oplus \mathcal{E}(-k) \xrightarrow{p} \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$. Since $h^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-k)\big) + h^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{E}(-k)\big) = h^0\big(X, \mathcal{O}_X\big) = 1$, it follows that $k \geq 1$, $h^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-k)\big) = 0$ and $h^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{E}(-k)\big) = 1$. Thus one can split both i and p through $\mathcal{E}(-k)$; hence $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k) \oplus \mathcal{E}_0$, where $\operatorname{rk}\mathcal{E}_0 = 2$ and $c_1(\mathcal{E}) = c_1(\mathcal{E}_0) + k$. One has an embedding $\mathbb{P}_0 := \mathbb{P}(\mathcal{E}_0) \subseteq \mathbb{P}$, and $\mathcal{O}_{\mathbb{P}}(\mathbb{P}_0) \cong \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-k)$. Since $X_0 := \mathbb{P}_0 \cap X$ is effective and $\mathcal{O}_X(X_0) \cong \mathcal{O}_{\mathbb{P}|X} \cong \mathcal{O}_X$, we see that $X_0 = \emptyset$, and sequence (2.1) becomes

$$0 \to \pi_0^* \det \mathcal{E}(-4) \to \pi_0^*\mathcal{F}(-2) \to \mathcal{O}_{\mathbb{P}_0} \to 0, \tag{7.1.1}$$

where $\pi_0 := \pi_{|\mathbb{P}_0}$. Twisting (7.1.1) by $\mathcal{O}_{\mathbb{P}_0}(2)$ and taking its direct image via π_0, we get

$$0 \to \mathcal{F} \to \mathcal{S}^2\,\mathcal{E}_0 \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k) \to 0. \tag{7.1.2}$$

In particular, $3c_1(\mathcal{E}_0) = c_1(\mathcal{S}^2\,\mathcal{E}_0) = c_1(\mathcal{F}) + k = c_1(\mathcal{E}) + k = c_1(\mathcal{E}_0) + 2k$, i.e. $c_1(\mathcal{E}_0) = k$ and $c_1(\mathcal{E}) = c_1(\mathcal{F}) = 2k$.

For each line $r \subseteq \mathbb{P}^n_{\mathbb{C}}$ the scheme $\varrho^{-1}(r)$ is connected (theorem 7.1 of [Jo]). Since we wish to deal with $\mathcal{E}_{|r}$ and $\mathcal{F}_{|r}$ we can assume that $n = 1$. One has $\mathcal{E}_0 \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k_2)$, $k_1 \geq k_2 \geq 1$ (X is connected) and $k_1 + k_2 = k$; then $k > k_1 \geq 1$ (this completes the proof of i)). Let $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h_2)$, $h_1 \geq h_2$. Sequence (7.1.2) becomes

$$0 \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h_2) \to \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2k_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2k_2) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k) \xrightarrow{a} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k) \to 0. \tag{7.1.3}$$

The matrix of a is

$$\big(a_{2k_1-k}, a_{2k_2-k}, a_0\big)\,,$$

where $a_i \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(i)\big)$. If $a_0 \neq 0$, then (7.1.3) splits, and ii) is proved.

If $a_0 = 0$ there is a monomorphism $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(k) \rightarrowtail \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(h_2)$. In particular $h_1 \geq k = k_1 + k_2$; hence $h_1 \geq 2k_2$. If $h_1 = 2k_2$ then $h_2 = 2k - h_1 = 2k - 2k_2 = 2k_1$.

Assume $h_1 > 2k_2$, and let $w \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{E}(-k)\big) \cong H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-k)\big)$, $v \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{E}(-k_1)\big) \cong H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-k_1)\big)$ and $u \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{E}(-k_2)\big) \cong H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-k_2)\big)$ be independent sections. Then $\mathbb{P}_0 = V_+(w) \subseteq \mathbb{P}$, and $X_0 := \mathbb{P}_0 \cap X \subseteq \mathbb{P}$ is defined by a system of equations of the form

$$\begin{aligned} w &= \alpha_{2k_2-h_1}u^2 + \alpha_{k_1+k_2-h_1}uv + \alpha_{2k_1-h_1}v^2 \\ &= \alpha_{2k_2-h_2}u^2 + \alpha_{k_1+k_2-h_2}uv + \alpha_{2k_1-h_2}v^2 = 0 \end{aligned}$$

where $\alpha_i \in H^0\big(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(i)\big)$. Since $h_1 > 2k_2$, then $\alpha_{2k_2-h_1} = 0$. If $2k_2 \neq h_2$, there is $y \in \mathbb{P}^1_{\mathbb{C}}$ such that $\alpha_{2k_2-h_2}(y) = 0$ and $(y, [1,0,0]) \in X_0 \neq \emptyset$, which is absurd. We conclude that $h_1 = 2k_1$, $h_2 = 2k_2$.

iii) is an easy computation from sequence (7.1.2). $\square$

Corollary 7.2. *Let ϱ, X, n be as above. If either $\mathcal{F}$ or $\mathcal{E}_0$ is uniform, then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k_1) \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k_2) \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k_1+k_2)$, $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(2k_1) \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(2k_2)$ and $\omega_{X|\mathbb{P}^n_{\mathbb{C}}} \cong \varrho^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k_1+k_2)$.*

Moreover, the natural map $\varrho_i^*\colon H^i\big(\mathbb{P}^n_{\mathbb{C}}, \mathbb{C}\big) \to H^i\big(X, \mathbb{C}\big)$ is an isomorphism for $0 \leq i \leq n-1$ and a monomorphism (but not an isomorphism) for $i = n$.

Proof. If $n \geq 5$ and either $\mathcal{F}$ or $\mathcal{E}_0$ is uniform, then the other one is also uniform and hence they both split (see [O–S–S], theorem I 3.2.3).

Now fix k_1 and k_2. Covers $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ with X smooth and invariants $\mathcal{E}$ and $\mathcal{F}$ correspond to sections η of a suitable open set $\mathcal{U} \subseteq H^0\big(\mathbb{P}^n_{\mathbb{C}}, \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}\big)$ (see Theorems 2.4 and 2.5). We claim that $\mathcal{U} \neq \emptyset$. Indeed, let $\alpha\colon Y \to \mathbb{P}^n_{\mathbb{C}}$ and $\beta\colon X' \to Y$ be two covers of degree 2 defined by general sections in $H^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k_2)\big)$ and $H^0\big(Y, \alpha^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k_1)\big)$ respectively, so that one can assume both Y and X' smooth. It is easy to check that $\varrho' := \alpha \circ \beta\colon X' \to \mathbb{P}^n_{\mathbb{C}}$ corresponds to a section $\eta' \in \mathcal{U}$. Obviously $\mathcal{U}$ is connected, and there is a smooth family

$$\mathcal{X} := \bigcup_{\eta \in \mathcal{U}} \big(D_0(\Phi_4(\eta)), \eta\big) \subseteq \mathbb{P} \times_{\mathbb{C}} \mathcal{U} \xrightarrow{\chi} \mathcal{U}$$

whose special fibres are X and X'. In particular $b_i(\chi^{-1}(\eta))$ is constant on $\mathcal{U}$; hence it suffices to prove that $b_i(\mathbb{P}^n_{\mathbb{C}}) = b_i(X')$, $0 \leq i \leq n-1$, and $b_n(\mathbb{P}^n_{\mathbb{C}}) < b_n(X')$, since in any case ϱ_i^* are monomorphisms because ϱ is finite. Since both α and β are cyclic covers with ample branch loci, one has isomorphisms $H^i\big(\mathbb{P}^n_{\mathbb{C}}, \mathbb{C}\big) \xrightarrow{\alpha_i^*} H^i\big(Y, \mathbb{C}\big) \xrightarrow{\beta_i^*} H^i\big(X', \mathbb{C}\big)$ for $0 \leq i \leq n-1$ (see [La], Theorem 2.1). Finally, $b_n(\mathbb{P}^n_{\mathbb{C}}) = b_n(X')$ if and only if both α_i^* and β_i^* are isomorphisms. Then, from theorem 2.7 of [Wi], we get that Y would be a hyperquadric in $\mathbb{P}^{n+1}_{\mathbb{C}}$ with n odd. But now it follows from theorem 2.9 of [Wi] that β_n^* can never be an isomorphism. $\square$

Remark 7.3. Let $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ be a cover of degree $d \leq n-1$ with X integral and smooth. Then there is $k \geq 2$ such that $\omega_{X|\mathbb{P}^n_{\mathbb{C}}} \cong \varrho^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k)$. As in the proof of Proposition 7.1 one obtains that $\mathcal{E} \cong \mathcal{E}_0 \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(k)$, where $\mathcal{E}_0$ is ample and $\operatorname{rk}\mathcal{E}_0 = d-2$. Since $X \cap \mathbb{P}(\mathcal{E}_0) = \emptyset$, the embedding $X \hookrightarrow \mathbb{P}$ gives rise to $X \hookrightarrow \mathbb{V} := \mathbb{V}(\mathcal{E}_0)$; hence the natural maps $\varrho_i^*\colon H^i\big(\mathbb{P}^n_{\mathbb{C}}, \mathbb{C}\big) \to H^i\big(X, \mathbb{C}\big)$ are isomorphisms if $i \leq n+2-d$. This slightly improves Theorems 1 and 2.1 of [La].

Remark 7.4. Of course, if Hartshorne's conjecture holds, the uniformity condition above is superfluous.

We now apply the results proved above in order to get a description of covers of $\mathbb{P}^n_{\mathbb{C}}$ with $R_{planar}(\varrho) = \emptyset$. In particular we will obtain the "local–global" criterion 1.3. We will make use of the theorems proved in [Fj].

Lemma 7.5. *Let $\varrho\colon X \to Y$ be a Gorenstein cover of degree 4 with $R_{planar}(\varrho) = \emptyset$. Assume that ϱ is not étale, Y is integral, smooth and projective over $\mathbb{C}$ with $\dim(Y) \geq 4$, $\Delta(X)$ is irreducible, every non–zero effective divisor on Y is ample, and for each cycle ζ of codimension 2 on Y one has $\zeta^2 \cdot c_1(\mathcal{E})^{n-4} \geq 0$. Then $\mathcal{F} \cong \mathcal{L} \oplus \mathcal{L}^2$ for some ample invertible $\mathcal{O}_Y$–sheaf $\mathcal{L}$.*

Proof. Proposition 4.4 and Remark 4.3 imply that $\Delta(\varrho)\colon \Delta(X) \to Y$ is a Gorenstein cover of degree 3 and the map $\Delta(\eta)\colon\ \mathcal{S}^3\check{\mathcal{F}} \to \det\check{\mathcal{F}}$ is surjective. It follows that we can imitate word for word the proof of [Fj], theorem 2.2 and corollary 2.3, taking into account that $c_1(\mathcal{E}) = c_1(\mathcal{F})$. In particular, either $\Delta(\varrho)$ is étale (and the same is true for ϱ by Proposition 4.4), or $\mathcal{F} \cong \mathcal{L} \oplus \mathcal{L}^2$ for some ample $\mathcal{L} \in \mathrm{Pic}(Y)$ (see [Fj], proof of theorem 2.1; see also [La]). $\square$

We now prove a more precise form of Theorem 1.3 which was stated in the introduction.

Theorem 7.6. *Let $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ be a cover of degree 4, with X smooth and $n \geq 5$.*

If $R_{planar}(\varrho) \neq \emptyset$, then $\mathrm{codim}_X\big(R_{planar}(\varrho)\big) \leq 4$ and the branch locus B_ϱ of ϱ has at least a fourfold point.

$R_{planar}(\varrho) = \emptyset$ if and only if there is $h > 0$ such that X is a quadrisecant of the line bundle associated to $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(h)$. In particular, $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(2h) \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(4h)$, $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(h) \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(2h) \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(3h)$, and $\omega_{X|\mathbb{P}^n_{\mathbb{C}}} \cong \varrho^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(3h)$.

Proof. The first assertion follows from Remark 4.3 and section 3.

Obviously, if X is a quadrisecant of a line bundle, then $R_{planar}(\varrho) = \emptyset$. Conversely, let $R_{planar}(\varrho) = \emptyset$. Then the structure of the sheaves $\mathcal{E}$, $\mathcal{F}$ and $\omega_{X|\mathbb{P}^n_{\mathbb{C}}}$ follows from Corollary 7.2 if we prove that $\mathcal{F} \cong \mathcal{L} \oplus \mathcal{L}^2$ for some $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}})$. Assume that $\Delta(X)$ is irreduciblel then we can apply Lemma 7.5.

It remains only to study the case when $\Delta(X)$ is reducible. In this case by Proposition 4.5 ii) there is an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(a) \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(b) \to 0$. Since $H^1\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(a-b)\big) = 0$, then $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(a) \oplus \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(b)$ with, for example, $a \leq b$. Thus, $\Delta(X)$ is represented in $\overline{\mathbb{P}}$ by the single equation $\alpha_{2a-b}s^3 + \alpha_a s^2 t + \alpha_b st^2 + \alpha_{2b-a}t^3 = 0$, where $s \in H^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}(-a)\big) \cong H^0\big(\overline{\mathbb{P}}, \mathcal{O}_{\overline{\mathbb{P}}} \otimes \overline{\pi}^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-a)\big)$, $t \in H^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{F}(-b)\big) \cong H^0\big(\overline{\mathbb{P}}, \mathcal{O}_{\overline{\mathbb{P}}} \otimes \overline{\pi}^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-b)\big)$ are independent sections and $\alpha_i \in H^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(i)\big)$. Since $n \geq 5$, if $b \neq 2a$ there is $y \in \mathbb{P}^n_{\mathbb{C}}$ such that $\alpha_i(y) = 0$, which contradicts the assumption $R_{planar}(\varrho) = \emptyset$.

Let $u \in H^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{E}(-h)\big)$, $v \in H^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{E}(-2h)\big)$ be independent sections. Up to a suitable linear transformation on the variables u and v, the above discussion implies that $X \subseteq \mathbb{P}$ has equations

$$\alpha(u,v) := \alpha_0 u^2 + \alpha_{3h}v + \alpha_{4h},$$
$$\beta(u,v) := \beta_0' u + \beta_0'' v^2 + \beta_h v + \beta_{2h},$$

where $\alpha_i, \beta_i \in H^0\big(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(i)\big)$. Obviously $\beta_0'' \neq 0$. If $\beta_0' = 0$ then

$$\emptyset \neq \{y \in \mathbb{P}^n_{\mathbb{C}}|\ \alpha_{3h}(y) = \alpha_{4h}(y) = \beta_h(y) = \beta_{2h}(y) = 0\} \subseteq \varrho(R_{planar}(\varrho));$$

hence we can assume $\beta_0' \neq 0$. Using β, we can write u explicitly as a quadratic polynomial in the variable v. Substituting in α, we get X as a quadrisecant of the line bundle associated to $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(h)$. $\square$

Remark 7.7. In [G–L] the loci $R_\ell(\varrho) := \{x \in X|$ the local degree at x is $e_\varrho(x) \geq \ell + 1\}$ are defined. In particular, $R_{planar}(\varrho) \subseteq R_3(\varrho)$. In theorem 1 of [G–L] it is proved that $\mathrm{codim}_X(R_\ell(\varrho)) \leq \ell$. In our setting, $R_\ell(\varrho) \neq \emptyset$ if $\ell \leq 3$.

Example 7.8. The bound on n is sharp. Let us consider a globally generated invertible sheaf $\mathcal{L}$ on $\mathbb{P}^n_{\mathbb{C}}$ with $n \leq 4$, and define $\mathcal{F} \cong \mathcal{L} \oplus \mathcal{L}^2$, $\mathcal{E} \cong \mathcal{L}^{\oplus 3}$. Since $\mathcal{H} := \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}$ is globally generated, for any general $\eta \in H^0\big(Y, \mathcal{H}\big)$ one has a smooth subscheme $X := D_0(\Phi_4(\eta)) \subseteq \mathbb{P}$. Moreover, any such X has global equations of the form

$$\alpha uv + \beta uw + \gamma v^2 + \delta vw + \epsilon w^2 = u^2 - vw = 0;$$

hence any such η has the right codimension at each point $y \in \mathbb{P}^n_{\mathbb{C}}$, i.e. $\varrho := \pi_{|X} : X \to \mathbb{P}^n_{\mathbb{C}}$ is a cover of degree 4 with $R_{planar}(\varrho) = \emptyset$.

We cannot hope to show that each cover $\varrho\colon X \to \mathbb{P}^n_{\mathbb{C}}$ of degree 4 is a quadrisecant of some line bundle as for covers of degree 3 with $n \gg 0$ (see [Fj] or [La]).

Example 7.9. Let Y, $\mathcal{L}$, $\mathcal{E}$, $\mathcal{F}$ and u, v, w be as in Example 4.6. $\mathcal{H} := \check{\mathcal{F}} \otimes \mathcal{S}^2\,\mathcal{E}$ is globally generated; hence $X := D_0(\Phi_4(\eta)) \subseteq \mathbb{P}$ is smooth for any general section $\eta \in H^0\big(Y, \mathcal{H}\big)$. Up to a suitable linear transformation of the coordinates u, v and w, $X \subseteq \mathbb{P}$ has global equations

$$u^2 + \alpha vw + \beta w^2 = v^2 + \gamma uw + \delta w^2 = 0,$$

where $\alpha, \gamma \in H^0\big(Y, \mathcal{L}\big)$, $\beta, \delta \in H^0\big(Y, \mathcal{L}^2\big)$. In particular, $X_w := X \cap V_+(w) = \emptyset$; hence $\varrho := \pi_{|X} : X \to Y$ is a cover of degree 4 with X smooth for each value of $\dim(Y)$. On the other hand, it cannot be a quadrisecant of any line bundle, at least when $Y \cong \mathbb{P}^n_{\mathbb{C}}$ with $n \geq 4$.

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